

On Projective Representations of Finite Wreath Products

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Abstract. The theory of induced projective representations is applied to finite wreath products, yielding algorithms which add to the collection of groups for which projective representations can be computed systematically. For finite Abelian and Abelian-wreath-cyclic groups, the factor sets are determined explicitly by establishing a one-to-one correspondence between certain lower triangular matrices and the inequivalent factor sets of these two classes of groups. This correspondence is used to determine the number and degrees of the inequivalent, irreducible projective representations.

1. Introduction. The problem of determining a complete set of inequivalent irreducible linear (ordinary) representations of a finite group is difficult unless the group is extremely uncomplicated; this has been discussed in a number of works, including the paper by Dixon [5], for example. The case of projective representations is all the more difficult. In this paper we add to the list of groups for which projective representations can be computed systematically by applying the theory of induced projective representations to finite wreath products. The corresponding application for linear representations has been considered by several authors; see, in particular, Kerber [9], [10] and Durbin [6].

In Section 2 we describe in detail an algorithm for constructing explicitly the projective representations of wreath products, including several remarks concerning computational efficiency. In Section 3 the factor sets of Abelian and Abelian-wreath-cyclic groups are derived in terms of matrices. Expressing these maps as arrays is a crucial step in programming the algorithm. This correspondence is further used to obtain information about the number and degrees of the irreducible projective representations. In Section 4, we illustrate the procedure for the specific group $Z_3 \sim Z_3$.

Groups are assumed finite and representations are over the field of complex numbers. We discuss only standard wreath products; this reduces notational difficulties but still brings out the essential ideas. The symbol \sim denotes equivalence of representations. In addition to the various specific references we mention the general references [4], [8] and [11].

Let A and B denote groups, and $K = A^B$ the set of functions from B to A , made into a group by componentwise multiplication. For $k \in K$ and $b \in B$, define $k^b \in K$ by $k^b(y) = k(yb^{-1})$ for all $y \in B$. Then, for each $b \in B$, the mapping $k \mapsto k^b$ is an automorphism of K , and the extension of K by the group of all such

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automorphisms is the *wreath product of A and B*, $A \sim B$. We identify K and B with the obvious subgroups of $A \sim B$. Then $K \triangleleft A \sim B$, $K \cap B = \{e\}$, and $KB = A \sim B$; typical elements will be denoted by kb .

2. Construction of Projection Representations of $A \sim B$. We describe an algorithm for constructing a complete set of inequivalent, irreducible projective representations of $A \sim B$, based on the method of induced representations, following Mackey [12] and Backhouse [1].

I. We first express a factor set of $A \sim B$ in terms of the following theorem:

THEOREM 1. *Every factor set ω' of $A \sim B$ is equivalent to a factor set ω of $A \sim B$ which may be uniquely represented in the form*

$$(*) \quad \omega(k_1 b_1, k_2 b_2) = \gamma(k_1, b_1 k_2 b_1^{-1}) \beta(b_1, k_2) \alpha(b_1, b_2),$$

where γ is a factor set of K , α is a factor set of B , and $\beta: B \times K \rightarrow T$ (the circle group) satisfies:

$$(i) \quad \beta(b, k_1 k_2) = \beta(b, k_1) \beta(b, k_2) \gamma(b k_1 b^{-1}, b k_2 b^{-1}) / \gamma(k_1, k_2)$$

and

$$(ii) \quad \beta(b_1 b_2, k) = \beta(b_1, b_2 k b_2^{-1}) \beta(b_2, k).$$

Moreover, for every choice of γ , α , and β satisfying (i) and (ii), the function ω defined by (*) is a factor set of $A \sim B$.

The results in Section 3 are useful in generating a complete set of inequivalent factor sets of $A \sim B$ decomposed according to Theorem 1.

II. The next step is to construct a complete set of inequivalent irreducible ω -representations of $A \sim B$ for the factor set ω decomposed in terms of the factor sets γ of K , α of B , and the function β .

Let D be an irreducible γ -representation of K of degree d . We associate with D two subgroups of $A \sim B$: the *inertia group* $H_D = \{x \in A \sim B: D^x \sim D\}$, where for all $k \in K$

$$D^x(k) = (\omega(x^{-1}, kx) \omega(k, x) / \omega(x^{-1}, x)) \cdot D(x^{-1} k x),$$

and the *inertia factor* $H_D^* = \{b \in B: D^b \sim D\}$. We note that the x -conjugate D^x of D is defined so that it has the same factor set γ as D . It is easily shown that H_D is the semidirect product of K and H_D^* . To extend D to a representation of H_D , we use the fact that for each $b \in H_D^*$ there exists a nonsingular matrix, say T_b , such that $D^b(k) = T_b^{-1} D(k) T_b$ for all $k \in K$. The irreducibility of D implies that the map $b \mapsto T_b$ is a λ -representation of H_D^* for some factor set λ of H_D^* . To find the equivalence operators T_b , it is sufficient to consider only generators of K and B , thus yielding a smaller set of equations. Furthermore, Backhouse [1] gives a formula for computing T_b . To determine the factor set λ of T , we need only compare a corresponding entry in the matrices $T_{b_1 b_2}$ and $T_{b_1} T_{b_2}$ for all b_1, b_2 in H_D^* .

Let $\{V_j: 1 \leq j \leq n\}$ be a complete set of inequivalent, irreducible $\alpha\lambda$ -representations of H_D^* . The set $\{U_j: 1 \leq j \leq n\}$, where $U_j: H_D \rightarrow GL(d, C)$ is defined by

$$U_j(kb) = D(k)T_b \otimes V_j(b),$$

is a set of ω -representations of H_D . By induction we lift this set to a set of irreducible ω -representations $\{U_j \uparrow A \sim B\}$ of $A \sim B$. Again, care must be taken to preserve factor sets; see [2].

To obtain a complete set of inequivalent ω -representations of $A \sim B$ we must consider orbits of the γ -representations of K . The *orbit* of the irreducible γ -representation D of K is the set

$$\begin{aligned} \mathcal{O}_D = \{D' : D' \text{ is an irreducible } \gamma\text{-representation of } K \\ \text{and } D' \sim D^b \text{ for some } b \in B\}. \end{aligned}$$

Then a complete set of inequivalent, irreducible γ -representations of K is the union of disjoint orbits, say $\mathcal{O}_{D_1} \cup \dots \cup \mathcal{O}_{D_m}$. By carrying out the method given above for each D_i , $1 \leq i \leq m$, we obtain

$$\{U_{ij} \uparrow A \sim B : 1 \leq i \leq m, 1 \leq j \leq n_i\},$$

a complete set of inequivalent, irreducible ω -representations of $A \sim B$.

As a final remark, this step begins with a γ -representation of K . To construct such a representation, the algorithm outlined in I and II is applicable. We note that for a direct sum the steps simplify considerably.

III. We obtain all irreducible inequivalent projective representations of $A \sim B$ by repeating I and II for each factor set in a complete set of inequivalent factor sets of $A \sim B$.

3. Abelian-Wreath-Cyclic Groups. We now apply the preceding algorithm to obtain specific information about the irreducible projective representations of finite Abelian and Abelian-wreath-cyclic groups. We prove that the factor sets of these two classes of groups can be explicitly described in terms of matrices. This expression enables us to determine the number and degrees of the inequivalent, irreducible projective representations.

Let A be finite Abelian, say $A = Z_{n_1} \oplus \dots \oplus Z_{n_s}$ with $n_i | n_{i+1}$ for $1 \leq i \leq s - 1$, and B finite of order t . We first consider the base group of $A \sim B$. If $\langle a_i \rangle = Z_{n_i}$, then $\{a_i : 1 \leq i \leq s\}$ generates A and K has a set of generators $\{x_1, \dots, x_{st}\}$, where x_l corresponds to a_i for $i \equiv l \pmod s$. Thus each element x of K can be expressed as

$$x = x_1^{u_1} x_2^{u_2} \dots x_{st}^{u_{st}},$$

$0 \leq u_i \leq n_i - 1$, where $i \equiv l \pmod s$. We identify x with the vector $(u_1, \dots, u_{st})^T$. We have the following theorem about the factor sets of K .

THEOREM 2. *Let G be the direct sum of m cyclic groups, say $G = G_1 \oplus \dots \oplus G_m$, with corresponding generators x_i of order n_i such that $n_i | n_{i+1}$, $1 \leq i \leq m - 1$. The elements of G are identified with m -tuples as described above. Let γ' be a factor set of G . Then γ' is equivalent to a factor set γ with a corresponding $m \times m$ lower triangular matrix G_γ such that*

$$\gamma(x, y) = \exp(-2\pi i x^T G_\gamma y).$$

The i, j th component $g_{i,j}$ of G_γ , for $i > j$, is an element of the set

$$C_{i,j} = \{l/N_{i,j} : l \in \mathbb{Z}, 0 \leq l \leq N_{i,j} - 1, N_{i,j} = (n_i, n_j)\}.$$

Furthermore, if the values of the i, j th entries range independently over $C_{i,j}$ for $1 \leq j < i \leq m$, then the G_γ 's correspond to a complete set of inequivalent factor sets of G .

Operations involving these matrices are done modulo 1. In order to prove this theorem and several others, we need the following fact proved by Backhouse.

LEMMA (BACKHOUSE [1, p. 278]). *Let ω be a factor set of an Abelian group G . Then ω is equivalent to the trivial factor set if and only if ω is symmetric, that is, $\omega(x, y) = \omega(y, x)$ for all x and y in G .*

Now let us prove Theorem 2.

Proof. Let x, y , and z be elements of G . Let G_γ be an $m \times m$ lower triangular matrix with its i, j th entry from $C_{i,j}$ ($1 \leq j < i \leq m$). If $\gamma: G \times G \rightarrow C$ is defined by

$$\gamma(x, y) = \exp(-2\pi i x^T G_\gamma y),$$

then it follows that γ is a factor set of G .

We now must show that for two factor sets γ and δ with corresponding matrices G_γ and G_δ , if $G_\gamma \neq G_\delta$, then γ is not equivalent to δ . If $\gamma \sim \delta$, then the function $(x, y) \mapsto \gamma(x, y)/\delta(x, y)$ is equivalent to the trivial factor set of G . Thus, $G_\gamma - G_\delta$ must be a symmetric matrix and so $G_\gamma = G_\delta$.

Since the number $\prod |C_{i,j}|$ of distinct matrices is $\prod (n_i, n_j)$ for $1 \leq j < i \leq m$, which Žmud' [13, p. 5] proves to be the order of $M(G)$, we conclude that this is a complete set of inequivalent factor sets.

Two corollaries follow.

COROLLARY 1. *Let γ be a factor set of an Abelian group G such that for each x and y in G ,*

$$\gamma(x, y) = \exp(-2\pi i x^T G_\gamma y),$$

for some $m \times m$ matrix G_γ . Then γ is equivalent to the trivial factor set if and only if G_γ is a symmetric matrix.

COROLLARY 2. *Let $A = Z_{n_1} \oplus Z_{n_2} \oplus \dots \oplus Z_{n_s}$ with $n_i | n_{i+1}$ for $1 \leq i < s - 1$, and let B have order t . If G_γ is an $st \times st$ lower triangular matrix whose i, j th entry $g_{i,j} \in C_{i,j}$, where*

$$C_{i,j} = \{l/N_{i,j} : l \in \mathbb{Z}, 0 \leq l \leq N_{i,j} - 1, N_{i,j} = (n_{i_0}, n_{j_0}), \\ i_0 \equiv i \pmod s, j_0 \equiv j \pmod s\},$$

($1 \leq j < i \leq st$), then $\gamma: K \times K \rightarrow C$, defined for all x and y in K by

$$\gamma(x, y) = \exp(-2\pi i x^T G_\gamma y),$$

is a factor set of K . If the values of the i, j th entries range independently over $C_{i,j}$ for $1 \leq j < i \leq st$, then the G_γ 's correspond to a complete set of inequivalent factor sets of K .

Using Frucht's result [7], we are able to specify the number and degrees of the inequivalent, irreducible γ -representations of the base group for a given factor set γ in terms of the matrix G_γ .

THEOREM 3. *Let G be the direct sum of m cyclic groups $G_1 \oplus \dots \oplus G_m$ such that G_i is generated by x_i of order n_i and $n_i | n_{i+1}$, $1 \leq i \leq m - 1$. Let γ be a factor set of G with corresponding matrix G_γ . The number of inequivalent, irreducible γ -representations of G is the number n_γ of solutions of the form (u_1, \dots, u_m) to the following set of m congruences:*

$$(*) \quad \left\{ \begin{array}{l} u_2 g_{2,1} + u_3 g_{3,1} + u_4 g_{4,1} + \dots + u_m g_{m,1} \equiv 0 \pmod{1} \\ -u_1 g_{2,1} + u_3 g_{3,2} + u_4 g_{4,2} + \dots + u_m g_{m,2} \equiv 0 \pmod{1} \\ -u_1 g_{3,1} - u_2 g_{3,2} + u_4 g_{4,3} + \dots + u_m g_{m,3} \equiv 0 \pmod{1} \\ \vdots \\ -u_1 g_{m,1} - u_2 g_{m,2} - \dots - u_{m-1} g_{m,m-1} \equiv 0 \pmod{1}. \end{array} \right.$$

Proof. We note that the matrix of coefficients appearing in this set (*) is $G_\gamma - G_\gamma^T$.

Frucht [7] proved that the number of inequivalent, irreducible γ -representations of G is the order of the subgroup

$$\{x | \gamma(x, y) = \gamma(y, x) \text{ for all } y \text{ in } G\}.$$

Thus, we want to find necessary and sufficient conditions for $x = x_1^{u_1} \cdot \dots \cdot x_m^{u_m}$ to be in this subgroup.

Let $y = x_1^{v_1} \cdot \dots \cdot x_m^{v_m}$ be an element of G . Then

$$\begin{aligned} \gamma(x, y) = \exp[-2\pi i(u_2 g_{2,1} + u_3 g_{3,1} + \dots + u_m g_{m,1})v_1 \\ + (u_3 g_{3,2} + \dots + u_m g_{m,2})v_2 + \dots + (u_m g_{m,m-1})v_{m-1}], \end{aligned}$$

and

$$\begin{aligned} \gamma(y, x) = \exp[-2\pi i(v_2 g_{2,1} + \dots + v_m g_{m,1})u_1 \\ + (v_3 g_{3,2} + \dots + v_m g_{m,2})u_2 + \dots + (v_m g_{m,m-1})u_{m-1}]. \end{aligned}$$

If γ is symmetric with respect to x , then $\gamma(x, x_i) \doteq \gamma(x_i, x)$ for $1 \leq i \leq m$. This fact implies the following equations:

$$\begin{aligned} \exp[-2\pi i(u_2 g_{2,1} + u_3 g_{3,1} + \dots + u_m g_{m,1})] &= \exp[-2\pi i \cdot 0] = 1, \\ \exp[-2\pi i(u_3 g_{3,2} + \dots + u_m g_{m,2})] &= \exp[-2\pi i(g_{2,1} u_1)], \\ &\vdots \\ \exp[-2\pi i(u_{j+1} g_{j+1,j} + u_{j+2} g_{j+2,j} + \dots + u_m g_{m,j})] \\ &= \exp[-2\pi i(g_{j,1} u_1 + g_{j,2} u_2 + \dots + g_{j,j-1} u_{j-1})] \\ &\vdots \\ 1 &= \exp[-2\pi i(g_{m,1} u_1 + g_{m,2} u_2 + \dots + g_{m,m-1} u_{m-1})]. \end{aligned}$$

So we have the set (*) of m congruences as a necessary condition.

Furthermore, if $x = x_1^{u_1} \cdots x_m^{u_m}$ and (u_1, \dots, u_m) satisfies (*), then $\gamma(x, y) = \gamma(y, x)$ for all y in G ; that is, this condition is sufficient.

Therefore, for A Abelian, a complete set of factor sets of the base group K of $A \sim B$ can be described explicitly. In addition, the number and degree of the inequivalent, irreducible γ -representations of K for a given γ can be calculated in terms of this description.

The next question we must answer is this: For a given factor set γ of K , when does there exist a function $\beta: B \times K \rightarrow C$ so that $(k_1 b_1, k_2 b_2) \mapsto \gamma(k_1, b_1 k_2 b_1^{-1}) \cdot \beta(b_1, k_2)$ is a factor set of $A \sim B$? We call such a factor set γ compatible. The following theorem gives the answer for A Abelian and B cyclic.

THEOREM 4. *Let A be finite Abelian with torsion decomposition $Z_{n_1} \oplus \cdots \oplus Z_{n_s}$ and B cyclic of order t . Let γ be a factor set of K with corresponding matrix G_γ . A necessary and sufficient condition for γ to be compatible is that $CG_\gamma C^T - G_\gamma$ be symmetric, where C is the $st \times st$ matrix*

$$C = \begin{bmatrix} 0 & 0 & \cdot & \cdots & 0 & 1 \\ 1 & 0 & \cdot & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}^s.$$

Proof. Let b be a generator of B . We note that C is the matrix representing the action of b on K .

By Mackey's decomposition, γ is compatible if and only if there exists a function $\beta: B \times K \rightarrow T$ such that

- (i) $\beta(b^n, k_1 k_2) = \beta(b^n, k_1) \beta(b^n, k_2) \gamma(b^n k_1 b^{-n}, b^n k_2 b^{-n}) / \gamma(k_1, k_2)$,
- (ii) $\beta(b^n b^m, k) = \beta(b^n, b^m k b^{-m}) \beta(b^m, k)$, and
- (iii) $\beta(e_B, k) = \beta(b^n, e_K) = 1$,

for all k_1, k_2 , and k in K and for $0 \leq n, m \leq t - 1$.

We first prove that the condition is necessary. Suppose that γ is compatible. From (i), we see that the factor set θ of K defined by

$$\theta(k_1, k_2) = \gamma(bk_1 b^{-1}, bk_2 b^{-1}) / \gamma(k_1, k_2)$$

must be symmetric. The matrix corresponding to θ is $CG_\gamma C^T - G_\gamma$; thus, by Corollary 1 of Theorem 3, we have that θ is symmetric if and only if $CG_\gamma C^T - G_\gamma$ is symmetric. Hence the symmetry of $CG_\gamma C^T - G_\gamma$ is certainly a necessary condition for the compatibility of γ .

To prove that the symmetry of $CG_\gamma C^T - G_\gamma$ is sufficient, we construct a function β which satisfies (i), (ii), and (iii). We define $\beta: B \times K \rightarrow T$ by

- (1) $\beta(b, k) = \exp[-\pi i k^T (CG_\gamma C^T - G_\gamma) k]$, and
- (2) $\beta(b^n, k) = \prod_{l=0}^{n-1} \beta(b, b^l k b^{-l})$.

It follows that if $CG_\gamma C^T - G_\gamma$ is symmetric and if we define β in this manner, then γ is compatible.

In order to state the resulting corollary, we must introduce notation concerning the matrices $\{G_\gamma: \gamma \in M(K)\}$ for A, B, γ , and G_γ as given in Theorem 4. Consider the $st - 1$ subdiagonals below the main diagonal of G_γ . The subdiagonal whose first component is $g_{st-m+1,1}$ is referred to as the *complement* of the subdiagonal whose first component is $g_{m+1,1}$; that is, the complement of the m th subdiagonal is the $(st - m)$ th subdiagonal. We see the m th subdiagonal has $st - m$ entries and that its complementary $(st - m)$ th subdiagonal has m entries for $1 \leq m \leq st/2$ if st is even and $1 \leq m \leq (st - 1)/2$ if st is odd. If st is even, then the $st/2$ th subdiagonal is its own complement.

COROLLARY. *The compatible factor sets of the base group K of $A \sim B$ can be described in terms of their corresponding matrices (see Corollary 2 of Theorem 1).*

(1) *Assume s and t are odd. The corresponding matrix G_γ of a compatible factor set γ of K has the following form: each of the first $(st - 1)/2$ subdiagonals is determined by a cycle of s elements. The complementary subdiagonals are obtained by taking the negatives of the continuation and repetition of the s -cycle. That is, the m th subdiagonal consists of repetitions of the s -cycle $\{g_{m+1,1}, g_{m+2,2}, \dots, g_{m+s,s}\}$. If the last entry in the m th subdiagonal is $g_{m+i,i}$, then the complementary subdiagonal begins with $-g_{m+i+1,i+1}, \dots, -g_{m+s,s}$ and continues with negatives of the s -cycle.*

(2) *Assume s is even and t is odd. For a compatible factor set γ of K , the first $st/2 - 1$ subdiagonals and their complements are constructed as in (1). The $st/2$ th subdiagonal has as its s -cycle $\{g_{l+1,1}, g_{l+2,2}, \dots, g_{l+s/2,s/2}, -g_{l+1,1}, \dots, -g_{l+s/2,s/2}\}$, where $l = st/2$. Thus its s -cycle is determined by $s/2$ elements.*

(3) *Assume t is even. The first $st/2 - 1$ subdiagonals and their complements in the matrix G_γ for a compatible γ are formed as in (1). The $st/2$ th subdiagonal has as its s -cycle $\{g_{l+1,1}, g_{l+2,2}, \dots, g_{l+s,s}\}$, where $l = st/2$ and $g_{l+i,k} \in C_{l+i,i}$ such that $g_{l+i,i} = -g_{l+i,i}$.*

As an example, consider $(Z_4 \oplus Z_8) \sim Z_4$. Its base group has $4^{22}8^6$ inequivalent factor sets, 4^68 being compatible. The compatibles have corresponding matrices:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_1 & b_2 & a_1 & 0 & 0 & 0 & 0 & 0 \\ d_1 & c_2 & b_1 & a_2 & 0 & 0 & 0 & 0 \\ -c_2 & d_2 & c_1 & b_2 & a_1 & 0 & 0 & 0 \\ -b_1 & -c_1 & d_1 & c_2 & b_1 & a_2 & 0 & 0 \\ -a_2 & -b_2 & -c_2 & d_2 & c_1 & b_2 & a_1 & 0 \end{bmatrix}$$

where a_l, b_l , and c_l belong to their respective $C_{l,j}$ ($1 \leq l \leq 2$), $d_1 \in \{0, 2/4\}$, and $d_2 \in \{0, 4/8\}$.

As we have seen, the compatible factor sets of the base group can be explicitly determined. If γ is such a factor set, then γ may be extended to more than one factor set of $A \sim B$; that is, there may be more than one function satisfying (i), (ii), and

(iii) in the proof of Theorem 4 corresponding to γ . However, in the next theorem, we see that in certain cases, each compatible factor set need only be extended once to obtain a complete set of inequivalent factor sets of $A \sim B$.

THEOREM 5. *Let $A = Z_{n_1} \oplus \dots \oplus Z_{n_s}$ such that $n_i | n_{i+1}$, $1 \leq i \leq s - 1$, and let $B = Z_r$. Let c denote the number of compatible factor sets of the base group of $A \sim B$. If t is odd or if n_1 and t are even, then $c = |M(A \sim B)|$.*

Proof. Using Blackburn’s theorem [3] to calculate $|M(A \sim B)|$, we obtain:

for t odd,

$$|M(A \sim B)| = [n_1^{(s+3(s-1))} n_2^{(s-1+3(s-2))} \dots n_{s-1}^{(2+3)} n_s]^{(t-1)/2},$$

and for t even,

$$|M(A \sim B)| = 2^s [n_1^{(s+3(s-1))} n_2^{(s-1+3(s-2))} \dots n_{s-1}^{(2+3)} n_s]^{(t/2-1)}.$$

We determine c by straightforward computation of the number of values which can appear as components in a compatible matrix. When s and t are odd, $c = \prod |C_{i,j}|$ for $i > j$, $2 \leq i \leq (st + 1)/2$, and $1 \leq j \leq s$. If s is even and t odd,

$$c = \prod |C_{i,j}| \cdot \prod |C_{m+1,l}|,$$

and if n_1 and t are even,

$$c = \prod |C_{i,j}| \cdot 2^s,$$

where $i > j$, $m = st/2$, $2 \leq i \leq m$, and $1 \leq l \leq s/2$. In each case, $c = |M(A \sim B)|$.

4. Example. We apply these results to find the irreducible projective representations of $Z_3 \sim Z_3$.

The base group K of $Z_3 \sim Z_3$ has 3 compatible factor sets γ_1, γ_2 and γ_3 ; γ_2 , for example, corresponds to

$$\begin{bmatrix} 0 & 0 & 0 \\ 1/3 & 0 & 0 \\ 2/3 & 1/3 & 0 \end{bmatrix}.$$

Using Theorem 3, we see that K has exactly 27 inequivalent, irreducible (linear) γ_1 -representations of degree 1, 3 inequivalent, irreducible γ_2 -representations of degree 3, and 3 such γ_3 -representations of degree 3. We construct these by applying the algorithm given in Section 2.

The factor sets of K are extended to $Z_3 \sim Z_3$ via functions constructed in the proof of Theorem 4; let these new factor sets be ω_1, ω_2 , and ω_3 . By determining orbits and inertia factors, we find that a complete set of inequivalent, irreducible representations of $Z_3 \sim Z_3$ consists of 17 linear ω_1 -representations (9 of degree 1 and 8 of degree 3), 1 ω_2 -representation of degree 9, and 1 ω_3 -representation of degree 9. Furthermore, we can describe these representations explicitly. An irreducible ω_2 -representation is $T \uparrow Z_3 \sim Z_3$, where T is an γ_2 -representation of K of the form

$$T(x_1^{u_1} x_2^{u_2} x_3^{u_3}) = a^{u_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^2 \end{bmatrix}^{u_1} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{u_2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & a^{-1} \\ a^{-2} & 0 & 0 \end{bmatrix}^{u_3}$$

for $a = \exp(2\pi/3)$.

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